Physics Placement Exam: Classical Mechanics and Electromagnetism

27-Aug-24

Problem 1: Consider the motion of a point particle of mass m subject to the central potential

$$V(r) = -\frac{\alpha}{r} - k \log r\,, \qquad \alpha > 0\,, \quad k > 0\,. \tag{1}$$

(a) Show that there is a critical value L_c for the angular momentum, above which there are no circular orbits. Compute L_c.

(b) How many circular orbits are there for $L < L_c$?

(c) Sketch a plot of the effective potential, for $L > L_c$ and for $L < L_c$.

(d) In the two cases, discuss qualitatively the possible orbits as a function of their energy E. "Discuss qualitatively" means: characterize the orbits as bound or unbound — no need to provide more details or explicit formulas; just refer to plot features.

Problem 2: Consider two particles on a line with Lagrangian

$$L = \tfrac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \left(e^{x_2} + e^{-x_1} + e^{x_1 - x_2}\right). \tag{2}$$

The potential energy is minimized at $x_1 = x_2 = 0$. By expanding L to quadratic order in x, find the frequencies of small oscillations around this minimum.

Problem 3: Let x be the position of a particle on a line and p its canonically conjugate momentum, so the Poisson bracket of two functions A, B on phase space is $\{A, B\} = \partial_x A\,\partial_p B - \partial_p A\,\partial_x B$, and the Hamilton equations imply $\dot{A} = \{A, H\}$. Consider the Hamiltonian

$$H \equiv \frac{p^2}{2} + \frac{\lambda}{2x^2}\,. \tag{3}$$

and define moreover

$$D \equiv xp\,, \qquad K \equiv \frac{x^2}{2}\,. \tag{4}$$

(a) Check that the Poisson brackets of H, D and K take the form

$$\{D, H\} = c_1 H\,, \qquad \{K, H\} = c_2 D\,, \qquad \{K, D\} = c_3 K \tag{5}$$

where c_1, c_2, c_3 are λ-independent constants. Find these constants.

(b) Using the relations (5) and $\dot{A} = \{A, H\}$, find $H(t)$, $D(t)$, and $K(t)$ in terms of their $t = 0$ initial values H_0, D_0 and K_0. Then use this together with the definitions (4) to read off the solutions for $x(t)$ and $p(t)$ in terms of x_0 and p_0.

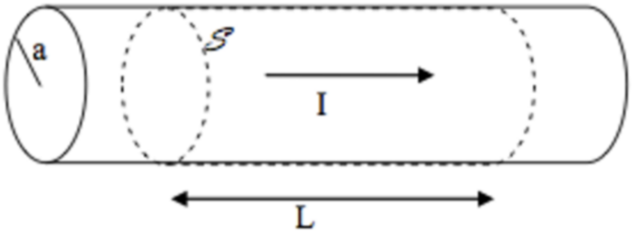
Problem 4: The potential on the surface of a sphere (radius R) is given by $V(r=R) = V_0 \cos 3\theta$.

a) Find the potential inside and outside the sphere.

b) What is the total charge of the sphere?

Hint: Use trigonometric identities to express the potential entirely in powers of $\cos\theta$

Problem 5:



Consider an infinite wire with cross sectional radius a and conductivity σ. A current I flows down the wire in the $\hat{z}$ direction. The electric field inside the wire is $\vec{E} = \left(\frac{I}{\pi a^2 \sigma}\right)\hat{z}$ and points along the direction of the current, while the magnetic field at the surface of the wire is $\vec{B} = \frac{\mu_0 I}{2\pi a}\hat{\phi}$ (here $\hat{\phi}$ wraps around the wire according to the right hand rule along the direction of I).

a) What is the magnitude and direction of the Poynting flux $\vec{S}$ at the surface of the wire?

b) Consider an imaginary closed cylindrical surface $\mathcal{S}$ that just encloses a wire (see above figure). The integral of the Poynting vector $\vec{S}$ over this surface is given by $\oint \vec{S} \cdot d\vec{A}$. What are the units of this integral? (given the simplest possible units) And what does this surface integral represent physically?

c) For the surface $\mathcal{S}$, it is possible to work out the value of the surface integral $\oint \vec{S} \cdot d\vec{A}$ in two different ways: (1) by computing it directly from $\vec{E}$ and $\vec{B}$; or (2) by knowing what the integral represents physically and simply writing down an expression for that thing.
Work out the value of $\oint \vec{S} \cdot d\vec{A}$ using <u>one</u> of these methods, and state which method you are using. **Your final answer must be in terms of just those variables given in the problem.**

Problem 6: A point charge q, of mass m, is attached to a spring of constant k. At time $t = 0$ it is given a kick, so its initial energy is $U_0 = \frac{1}{2}mv_0^2$. Now it oscillates, gradually radiating away this energy. Assume the radiation damping is small, so you can write the equation of motion as

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

and the solution as

$$x(t) = \frac{v_0}{\omega_0}\, e^{-\gamma t/2}\, \sin\omega_0 t \quad ,$$

with $\omega_0 = \sqrt{\frac{k}{m}}$, $\gamma = \omega_0^2\tau$ and $\gamma \ll \omega_0$ (drop γ^2 in comparison to ω_0^2, and when you average over a complete cycle, ignore the change in $e^{-\gamma t}$).

a) Determine the value of τ in terms of the other parameters in the problem.

b) Confirm that the total energy radiated is equal to U_0.

Potentially Useful Equations and Definitions

Trigonometric identities: $\cos(A{+}B) = \cos A \cos B - \sin A \sin B$, $\sin(A{+}B) = \sin A \cos B + \sin B \cos A$

Cylindrical coordinates: $x = s\cos\phi$, $y = s\sin\phi$, $z = z$, $s = \sqrt{x^2+y^2}$

$$\begin{aligned}
\nabla t &= \frac{\partial t}{\partial s}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\,\hat{\phi} + \frac{\partial t}{\partial z}\,\hat{\mathbf{z}} \\
\nabla\cdot\mathbf{v} &= \frac{1}{s}\frac{\partial(s v_s)}{\partial s} + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \\
\nabla\times\mathbf{v} &= \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial(s v_\phi)}{\partial s} - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}} \\
\nabla^2 t &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}
\end{aligned}$$

Spherical coordinates: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

$$\begin{aligned}
\nabla t &= \frac{\partial t}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\,\hat{\phi} \\
\nabla\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\ v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \\
\nabla\times\mathbf{v} &= \frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta\ v_\phi)}{\partial \theta} - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial(r v_\phi)}{\partial r}\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial(r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right]\hat{\phi} \\
\nabla^2 t &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}
\end{aligned}$$

Solutions when there is no ϕ dependence:

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l\, r^l + \frac{B_l}{r^{l+1}}\right] P_l(\cos\theta) \quad \text{satisfies} \quad \nabla^2\Phi = 0 \quad .$$

$$P_0(u) = 1\ ,\ P_1(u) = u\ ,\ P_2(u) = \frac{3}{2}u^2 - \frac{1}{2}\ ,\ P_3(u) = \frac{5}{2}u^3 - \frac{3}{2}u$$

$$\int_{-1}^{1} P_m(u)P_n(u)\,du = \frac{2}{2n+1}\,\delta_{m,n}$$

Electrostatic energy (W):

Discrete charges:

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j\neq i}^{n} \frac{q_i q_j}{r_{ij}}$$

Continuous charge distribution:

$$W = \frac{\epsilon_0}{2} \int \mathbf{E}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})\ d^3r = \frac{1}{2} \int \rho(\mathbf{r}) \Phi(\mathbf{r})\ d^3r$$

Field from magnetic dipole:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{3m} \cdot \hat{\mathbf{r}} - \mathbf{m}}{r^3}$$

Energy density and flux, momentum density:

$$u = \frac{1}{2}\epsilon_0\ \mathbf{E} \cdot \mathbf{E} + \frac{1}{2\mu_0} \mathbf{B} \cdot \mathbf{B} \quad , \quad \mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \quad , \quad \mathbf{g} = \epsilon_0 (\mathbf{E} \times \mathbf{B})$$

Speed of light, impedance of the vacuum:

$$\mu_0 \epsilon_0 = \frac{1}{c^2} \quad , \quad \mu_0 c = 377\ \Omega$$

Ohm's Law: $\vec{J} = \sigma \vec{E}$

Larmor formula:

$$P = \frac{2}{3} \frac{q^2}{4\pi\epsilon_0 c^3}\ a^2$$

Quantum Mechanics, Statistical Mechanics and Thermodynamics

28-Aug-24

Problem 1: Use the variational principle with a family of Gaussian trial wave functions ψ_a satisfying $\langle\psi_a|\psi_a\rangle = 1$, $\langle\psi_a|x^2|\psi_a\rangle = \frac{1}{2a}$, that is to say a family of trial wavefunctions given by

$$\psi_a(x) = \left(\frac{a}{\pi}\right)^{1/4} e^{-ax^2/2}\,, \tag{1}$$

to estimate, and bound, the ground state energy of the Hamiltonian

$$H = \frac{p^2}{2m} - V_0\,\delta(x)\,, \qquad V_0 > 0\,, \tag{2}$$

for a 1D particle with a δ-function potential, and compare your answer with the exact $E_0 = -\frac{mV_0^2}{2\hbar^2}$

Problem 2: The Hamiltonian a two-dimensional isotropic harmonic oscillator is

$$\hat{H}_0 = \tfrac{1}{2m}(\hat{p}_1^2 + \hat{p}_2)^2 + \tfrac{1}{2}m\omega^2(\hat{x}_1^2 + \hat{x}_2^2) = \hbar\omega\big(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\big)\,, \tag{3}$$

with the two expressions related by

$$\hat{x}_i = \sqrt{\tfrac{\hbar}{2m\omega}}\,(a_i + a_i^\dagger), \qquad \hat{p}_i = i\sqrt{\tfrac{\hbar m\omega}{2}}\,(a_i^\dagger - a_i)\,. \tag{4}$$

Applying an external force $F(t)$ in the x_1-direction amounts to adding a term $\hat{V}(t) = -F(t)\,\hat{x}_1$ to $\hat{H}_0$:

$$\hat{H}(t) = \hat{H}_0 - F(t)\,\hat{x}_1\,. \tag{5}$$

Assuming the oscillator is in its ground state in the far past $t \to -\infty$, and taking the force $F(t)$ to be a time-symmetric exponential pulse centered at $t = 0$,

$$F(t) = F_0\,e^{-\gamma\,|t|}\,, \tag{6}$$

with F_0 small enough to render leading-order perturbation theory (equation (9)) adequate, what is the probability of finding the oscillator in its first excited energy level in the far future $t \to +\infty$?

Problem 3: Consider a hydrogen atom in a strong uniform magnetic field B along the x_3-axis. Assume that the field is weak enough that the energy shifts due to B are much smaller than the energy gap between the ground state and first excited level of hydrogen, but strong enough that they are much greater than the fine-structure (relativistic kinetic energy and spin-orbit) corrections, and ignore the latter in this problem. That is to say, model this setup as a non-relativistic spin-$\frac{1}{2}$ particle (the electron) with Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{V}\,, \qquad \hat{H}_0 = \frac{\hat{p}^2}{2m_e} - \frac{e^2}{4\pi\epsilon_0}\,\frac{1}{\hat{r}}\,, \qquad \hat{V} = \frac{eB}{2m_e}\big(\hat{L}_3 + 2\,\hat{S}_3\big) \tag{7}$$

where $\hat{L}_3$ and $\hat{S}_3$ are the components of the electron orbital angular momentum and spin along the x_3-axis. The eigenstates of $\hat{H}_0$ are $|n, l, m_l, m_s\rangle$, with energy $E_n = -E_\star/n^2$, and l, m_l, m_s the electron orbital angular momentum and spin quantum numbers.

(a) Into how many different energy levels does the 1S level split?
(b) Into how many different energy levels does the 2P level split?
(c) Now consider the allowed electromagnetic (dipole) transitions from 2P states to 1S states. In the absence of a magnetic field, these would give a single line in the emission spectrum. How many lines are there when the magnetic field is present?

Problem 4: Consider a two-dimensional system consisting of a macroscopically large number N of non-interacting, identical spin-$\frac{1}{2}$ particles of mass m, trapped in a two-dimensional isotropic harmonic oscillator potential $V = \frac{1}{2}m\omega^2(x_1^2 + x_2^2)$, as in (3). Find the energy up to which the single-particle energy eigenstates are occupied when the N-particle system is in its ground state, a.k.a. the Fermi energy ϵ_F, and from this the size (radius) of the system at zero temperature.

(Note that because N is macroscopically large, meaning something like $N \gtrsim 10^{20}$ or so, counting the number of single-particle states with energy less than ϵ is very well approximated by calculating a volume, for the energy scales ϵ of interest.)

Problem 5: Cold interstellar molecular clouds often contain the molecule cyanogen (CN), whose first rotational excited states have an energy of $4.7 \times 10^{-4}\,\mathrm{eV}$ (above the ground state). There are actually 3 such excited states, all with the same energy. In 1941, studies of the absorption spectrum of starlight that passes through these molecular clouds showed that for every 10 CN molecules that are in the ground state, approximately 3 others are in the first excited states (that is, an average of 1 in each of these states for every 10 in the ground state). To account for this data, astronomers suggested that the molecules might be in thermal equilibrium with some "reservoir" with a well-defined temperature. Find this temperature.

Problem 6: Consider a system of N non-interacting particles, each of which can occupy just two energy eigenstates: a ground state $|0\rangle$ with energy $\epsilon_0 = 0$, and an excited state $|1\rangle$ with energy $\epsilon_1 > 0$. The system is in equilibrium with a heat reservoir at fixed temperature $T > 0$. What is the probability for the system to be in its ground state

(a) if the particles are distinguishable (so the N-particle states are $|m_1, \ldots, m_N\rangle$ with $m_i \in \{0, 1\}$)

(b) if the particles are indistinguishable bosons (so N-particle states are $|n_0, n_1\rangle$ with $n_0 + n_1 = N$)

in the limit $N \to \infty$?

Problem 7: You're on a long-distance flight having a cold drink and watching a romantic comedy, which inexplicably yet predictably makes you tear up a little. A teardrop ends up in your drink. What is approximately the increase in entropy of the tear+drink system due to the transfer of heat from tear to drink, and by what factor does the corresponding microstate degeneracy of the system increase?

This question is meant to be an estimate, but for concreteness, assume the specific heat capacity of all liquids involved equals that of water, $c_{\text{water}} = 4.2\ \mathrm{J\,/\,K\,ml}$, that a tear equals $0.05\,\mathrm{ml}$ (so its heat capacity is $C_{\text{tear}} = 0.21\,\mathrm{J/K}$), and that tear and drink temperatures are $T_{\text{tear}} = 310\,\mathrm{K}$, $T_{\text{drink}} = 275\,\mathrm{K}$.

Before plugging in numbers however, first obtain the result in general. Because the tear is much smaller than the drink, you can assume here that the drink's temperature remains constant at T_{drink} during tear-to-drink heat transfer, while the temperature of the tear drops from T_{tear} to T_{drink}. The problem amounts to calculating the change in entropy of tear + drink as a result of heat going from tear to drink.

Potentially Useful Formulas

Integrals:

$$
\begin{aligned}
\int dx\, e^{-ax^2} &= \sqrt{\frac{\pi}{a}} \\
\int dx\, x^2\, e^{-ax^2} &= -\partial_a \int dx\, e^{-ax^2} \\
\int dx\, x^{2n}\, e^{-ax^2} &= (-\partial_a)^n \int dx\, e^{-ax^2} \\
\int_{-\infty}^{\infty} dx\, e^{-a|x|}\, f(x) &= \int_0^{\infty} dx\, e^{-ax}\big(f(x) + f(-x)\big)
\end{aligned}
\tag{8}
$$

Time-dependent perturbation theory:

For a system with Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}(t)$ where $\hat{V}(t)$ is a time-dependent perturbation vanishing in the far past and future, i.e. $\hat{V}(\pm\infty) = 0$; denoting the eigenstates of $\hat{H}_0$ by $|\alpha\rangle$, i.e. $\hat{H}_0|\alpha\rangle = E_\alpha|\alpha\rangle$; expanding $|\psi(t)\rangle = \sum_\alpha c_\alpha(t)\, e^{-iE_\alpha t/\hbar}|\alpha\rangle$; if system in state $|\beta\rangle$ in the far past, i.e. $c_\alpha(-\infty) = \delta_{\alpha\beta}$, then

$$
c_\alpha(+\infty) = \delta_{\alpha\beta} \; - \; \frac{i}{\hbar}\int_{-\infty}^{+\infty} dt\, \langle\alpha|\hat{V}(t)|\beta\rangle \; + \; O(V^2) \tag{9}
$$

is the amplitude for the system to be in the state $|\alpha\rangle$ in the far future.

Harmonic oscillator

$$
\hat{H} = \omega\hbar(a^\dagger a + \tfrac{1}{2})\,, \quad [a, a^\dagger] = 1\,, \quad |n\rangle = \tfrac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle\,, \quad \langle n'|n\rangle = \delta_{n'n}\,, \quad \hat{H}\,|n\rangle = (n+\tfrac{1}{2})\omega\hbar\,|n\rangle\,. \tag{10}
$$

Constants:

Boltzmann's constant: $k = 1.38 \times 10^{-23}\mathrm{J/K} = 8.62 \times 10^{-5}\,\mathrm{eV/K}$.

Entropy of things

(11)

thing	$T\,(\mathrm{K})$	$S/k \sim$
2 TB harddrive	0	10^{13}
gram of DNA	0	10^{21}
helium balloon	300	10^{24}
cold gallon of water	280	10^{27}
hot gallon of water	320	10^{27}
cat	310	10^{27}
you	310	10^{28}
empire state building	275	10^{35}
earth	10^3	10^{51}
sun	10^7	10^{57}
solar mass BH horizon	10^{-7}	10^{77}
cosmic microwave background	2.7	10^{88}
M31 mass BH horizon	10^{-16}	10^{95}
universe horizon	10^{-30}	10^{122}