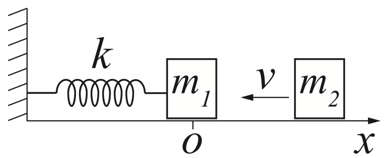
Physics placement exam:
Classical mechanics and electromagnetism

August 25, 2025

Please do all six problems. Your work should be organized and legible. A number of potentially useful formulae are collected on the last page of the exam. You are not allowed to use your phone, calculator, or any device with messaging capabilities during the exam.

Problem 1

A mass m_1 is attached to a wall by a linearly damped horizontal spring with spring constant k and friction coefficient γ (i.e., the friction force is $F = -\gamma\dot{x}$). It is sitting at rest with the spring unstretched when it is struck by a second mass, m_2, moving to the left with velocity v:



The collision is instantaneous and after the collision the masses stick together. Use the coordinate system shown, with origin at the end of the unstretched spring and positive x-displacement to the right. In the following, consider the masses to be equal, $m_1 = m_2 = m$.

a) Find the velocity of the combined masses immediately after collision.

b) What value of m is necessary such that after the collision the system is critically damped? Give your answer in terms of the constants k and γ. As a reminder, the 'critically damped' case is the threshold between oscillatory and non-oscillatory behavior.

c) Assume the collision occurs at time $t = 0$ and that the system is critically damped; what will be the position and velocity of the combined masses vs. time after the collision? Make sure to use initial conditions to eliminate integration constants. [*Hint: To solve the equation of motion in the critically damped case, try the ansatz $x(t)$ = polynomial $\times$ exponential.*]

Problem 2

A small bead of mass m slides without friction on a straight wire, inclined at angle α to the horizontal. The wire is pulled straight down towards the earth with a constant acceleration a, and without changing its orientation or deforming (see fig. 1).

a) Write down the Lagrangian for the system, using as a generalized coordinate the position of the bead along the wire, q. [*Hint: it may be useful to note that the lower end evolves in time as $(x, y) = (0, y_0 - \frac{1}{2}at^2)$).*]

b) From this Lagrangian, find an expression for the acceleration $\ddot{q}$ of the bead along the wire.

c) What value for the downward acceleration of the wire is necessary such that the bead slides *up* the wire (in the reference frame of the accelerating wire)?

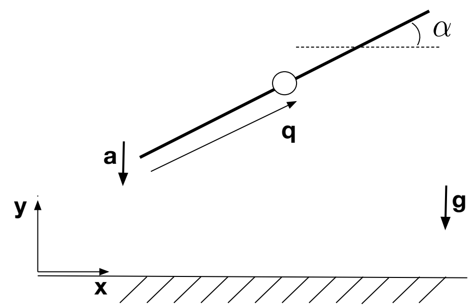


Figure 1: *Setup for Problem 2.*

Problem 3

A simple pendulum of mass m_2 and length l is connected to a mass m_1 at the point of support (see fig. 2). The mass m_1 can slide without friction on a horizontal line lying in the plane in which m_2 moves. The system is located in a uniform gravitational field with gravitational acceleration g pointed downward.

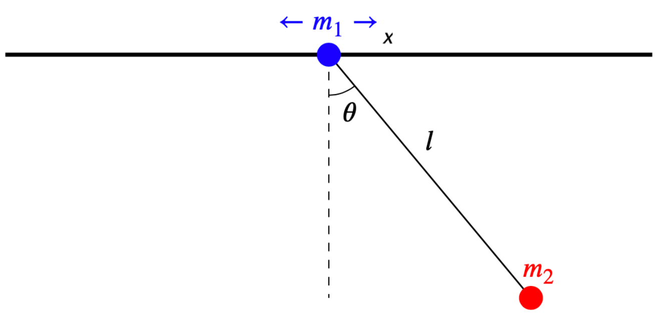


Figure 2: *Setup for Problem 3.*

a) Determine the Lagrangian for the system in terms of the horizontal coordinate x of mass m_1 and the angular coordinate θ of mass m_2.

b) Identify any cyclic coordinate(s) and determine the associated conserved quantity.

c) Set this conserved quantity to zero for simplicity and use the resulting equation to obtain a relation between x and θ.

d) Use the result of part 3 to express the cartesian coordinates of the mass m_2, (x_2, y_2), in terms of θ alone.

e) Show that (x_2, y_2) move along the arc of an ellipse. What are the semi-major and semi-minor axes of this ellipse? Show that in the limit $m_1 \to \infty$, the path becomes the arc of a circle, as expected for the motion of a simple pendulum with fixed support point.

Problem 4

The potential on the surface of a sphere (radius R) is given by $V(r=R) = V_0 \cos^2\theta$.

a) Find the leading term for $r \gg R$ in the expansion of the potential in powers of r.

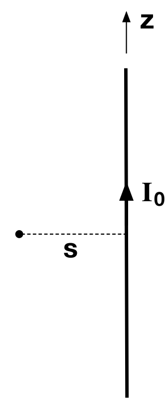
b) What is the total charge of the sphere?

Problem 5

The general expressions for the scalar and vector potentials are

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,',t_r)\,d^3r'}{|\vec{r}-\vec{r}\,'|} \qquad (1)$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)\,d^3r'}{|\vec{r}-\vec{r}\,'|}, \qquad (2)$$



where t_r is the retarded time.
A long (effectively infinite) neutral wire on the z-axis has zero current for $t < 0$. At $t = 0$ a steady current I_0 is suddenly turned on in the $+\hat{z}$ direction (see figure).

a) Consider a point at a distance s from the wire. At what time do the electric and/or magnetic fields first become non-zero at this point? Hereafter call this time t_s ('s' for when the field starts at position s).

b) What is the value of the scalar potential V at position s at time $t > t_s$?

c) What is the direction of the vector potential $\vec{A}$ at position s at time $t > t_s$?

d) What is the direction of the electric field $\vec{E}$ at position s at time $t > t_s$?

e) What is the direction of the magnetic field $\vec{B}$ at position s at time $t > t_s$?

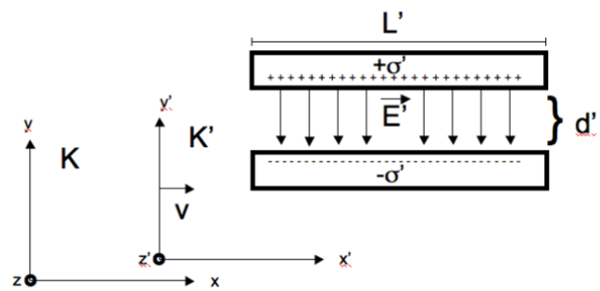
f) At times $t > t_s$, the magnitude of the vector potential can be written in this form:

$$A(s,t) = \frac{\mu_0}{4\pi}\int_{-??}^{+??} \frac{I_0}{\sqrt{s^2+z^2}}dz \qquad (3)$$

What are the appropriate limits of integration? Justify your answer carefully.

Problem 6

A pair of long conducting plates is at rest in the frame K'. In this frame the plates have length L' and are separated by a distance d', and the top and bottom plates possess a surface charge $\pm\sigma'$, establishing a uniform electric field $\vec{E}\,' = -\frac{\sigma'}{\epsilon_0}\hat{y}$ between the plates. The plates move to the right with velocity $\vec{v} = v\,\hat{x}$ with respect to a second frame K (see figure).



a) What is the length L of the plates as measured in the K frame? What is the separation d of the plates as measured in the K frame?

b) What is the surface charge density σ of the plates in the K frame as compared to σ'?

c) What is the electric field $\vec{E}$ between the plates as measured in the K frame?

d) Explain why there is now a magnetic field $\vec{B}$ between the plates in the K frame.

e) Determine the magnetic field $\vec{B}$ in the K frame.

Potentially useful formulae and definitions

- Form of solutions for an oscillator with damping:

$$\begin{aligned}
\text{damped oscillations:} \quad & x(t) = [A\sin(\omega t) + B\cos(\omega t)]\exp(-\Gamma t) \\
\text{aperiodic, damped:} \quad & x(t) = (A + Bt)\exp(-\Gamma t) \\
\text{strongly damped:} \quad & x(t) = A\exp(-\Gamma_1\, t) + B\exp(-\Gamma_2\, t)
\end{aligned}$$

- Solutions to Laplace equation in spherical coordinates (r, θ, ϕ) when there is no ϕ dependence:

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l\; r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos\theta)$$

$$P_0(u) = 1\ , \qquad P_1(u) = u\ , \qquad P_2(u) = \frac{3}{2}u^2 - \frac{1}{2}\ , \qquad P_3(u) = \frac{5}{2}u^3 - \frac{3}{2}u$$

$$\int_{-1}^{1} P_m(u)P_n(u)\; du = \frac{2}{2n+1}\;\delta_{m,n}$$

- Potentials:

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}\ , \qquad \vec{B} = \vec{\nabla}\times\vec{A}$$

Physics placement exam:

Quantum mechanics, statistical mechanics and thermodynamics

August 26, 2025

Please do all six problems. Your work should be organized and legible. A number of potentially useful formulae are collected on the last page of the exam. You are not allowed to use your phone, calculator, or any device with messaging capabilities during the exam.

Problem 1

A spin-1/2 particle has a magnetic moment $\vec{\mu} = \gamma \vec{S}$, where γ is a positive constant and $\vec{S}$ is the spin. The particle is placed in a uniform magnetic field $\vec{B}$ that points in the z-direction, so that the Hamiltonian describing the system is

$$H = -\vec{\mu} \cdot \vec{B} = -\gamma \vec{S} \cdot \vec{B} = -\frac{\gamma B \hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

a) What are the energy eigenvalues and corresponding spin wave functions?

b) Suppose that at time $t = 0$ the particle's spin wave function is

$$\chi = \begin{pmatrix} \sin\beta \\ \cos\beta \end{pmatrix},$$

where β is a real number. What is the wave function at some later time t?

c) Given the initial state in b), what are the expectation values of S_x and S_z as a function of time?

Problem 2

Consider a harmonic oscillator in three dimensions, with $V(x, y, z) = \frac{1}{2} m\omega_1^2 (x^2 + y^2) + \frac{1}{2} m\omega_2^2 z^2$, where $\omega_2 = 2\omega_1$.

a) Find the three lowest energy eigenvalues. Determine the number of degenerate states for each of these.

b) Do any of the components of the angular momentum commute with the Hamiltonian? If yes, which components?

Problem 3

A 2×2 matrix is parameterized as $\frac{1}{2}(A\ \mathbb{1} + \vec{\sigma} \cdot \vec{B})$, where $\mathbb{1}$ is the identity matrix, A is a real number, $\vec{B}$ is 3-dimensional vector of real numbers, and $\vec{\sigma}$ represents the 3 Pauli spin matrices.

a) Find the conditions on A and $\vec{B}$ that would make this matrix a valid density matrix for a pure state. You may find the following identity (written using Einstein summation convention) useful: $\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i\epsilon_{ijk}\sigma_k$.

b) Assuming a pure state, and that the 2-state system is spin-1/2, and the matrix is in the z-representation, find the values of A and $\vec{B}$ that maximize the expectation value of $\langle \hat{S}_y \rangle$, the y-component of the spin operator.

c) What conditions on A and $\vec{B}$ are needed to represent the density matrix of a mixed state?

Problem 4

a) (3 points) Consider a probability distribution function given by

$$P(x) = \lambda e^{-\lambda x} \tag{1}$$

for $x \geq 0$, and $P(x) = 0$ for $x < 0$. Here, $\lambda > 0$ is a constant. Find the mean, the variance, and the standard deviation of the variable x. A useful integral is $\int_0^\infty x^n e^{-\lambda x} = n!/\lambda^{n+1}$, where n is a non-negative integer.

b) (2 points) Now consider a set x_1, x_2, ..., x_N, with each element independently drawn from $P(x)$ in question (a). Consider the variable $y = (x_1 + x_2 + ... + x_N)/N$. What are the mean, the variance, and the standard deviation of this variable?

c) (5 points) Consider a monoatomic ideal gas at temperature T. The Mawell-Boltzmann distribution of the velocity vectors of gas atoms $\vec{v} = (v_x, v_y, v_z)$ is given by

$$P(v_x, v_y, v_z) = \frac{1}{(2\pi)^{3/2} v_{\text{th}}^3} \exp\left[-\frac{\vec{v}^2}{2v_{\text{th}}^2}\right], \tag{2}$$

where

$$v_{\text{th}} = \sqrt{k_B T/m}. \tag{3}$$

Here k_B is the Boltzmann constant and m is the mass of the atom.

Write down the expression for the probability density function $P(E)$, where E is the kinetic energy of the atom. Write down the integral expression for the fraction of atoms with energy $E > 30\, k_B T$, and show that it does not depend on k_B, m, or T. Estimate this fraction in terms of purely numerical factors.

Problem 5

a) (5 points) Consider N free particles confined inside a large cube of volume V, with periodic boundary conditions on the particles' wavefuntions. The density of states in 3-dimensional $\vec{k}$-space is given by

$$g(\vec{k}) = \frac{V}{(2\pi)^3}. \tag{4}$$

Take the above equation as given (you don't need to prove it) and show that if the particles are spin-1/2 fermions (i.e., 2 particles of opposite spins are allowed in each $\vec{k}$-state), then the Fermi surface has the radius

$$k_F = (3\pi^2 n)^{1/3}, \tag{5}$$

where $n = N/V$. In the non-relativistic case,

$$\epsilon(k) = \frac{\hbar^2 k^2}{2m}, \tag{6}$$

where m and ϵ are the particle mass and energy, respectively. Assume $T = 0$ and show that in that case the degeneracy pressure P of the fermions scales as

$$P \propto n^{5/3}. \tag{7}$$

Find the coefficient of proportionality.

b) (5 points) Consider now the same Fermi gas at a small but non-zero temperature $T \ll \epsilon(k_F)/k_B$. Recall that only the particles with energies within $\sim k_B T$ from the Fermi energy $\epsilon(k_F)$ are excited and use this to argue that the heat capacity scales as

$$C \sim N k_B \frac{k_B T}{\epsilon(k_F)}. \tag{8}$$

You do not need to compute the numerical factor.

Problem 6

a) (5 points) Derive a partition function $Z(\beta, \omega)$ for a quantum harmonic oscillator of proper frequency ω immersed in a heat bath of temperature T; here $\beta = (k_B T)^{-1}$ and k_B is the Boltzmann constant. Furthermore, find the mean energy

$$E = -\frac{\partial \log Z}{\partial \beta}. \tag{9}$$

Show that for $T \gg \hbar\omega/k_B$, $E \simeq k_B T$.

b) (5 points) Consider an ensemble of N distinguishable 2-state quantum systems, with ϵ_1 and ϵ_2 being the energies of the two states. Assume that the ensemble is in thermal contact with a heat bath of temperature T, and find the heat capacity C of the ensemble as a function of T. Give your answer in terms of the dimensionless quantities $\tilde{C} \equiv C/(N k_B)$ and $\tilde{T} \equiv k_B T/(\epsilon_2 - \epsilon_1)$.

Potentially useful formulae and definitions

- Spin operators and commutators:

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z \ , \qquad [\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x \ , \qquad [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y$$

  Spin 1/2:

$$\hat{S}_x = \frac{\hbar}{2}\hat{\sigma}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{S}_y = \frac{\hbar}{2}\hat{\sigma}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Geometric series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$