Columbia University
Department of Physics
QUALIFYING EXAMINATION

Monday, January 11, 2016
1:00PM to 3:00PM
Classical Physics
Section 1. Classical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

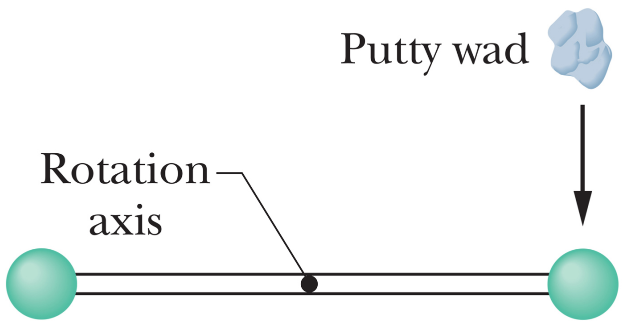
You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

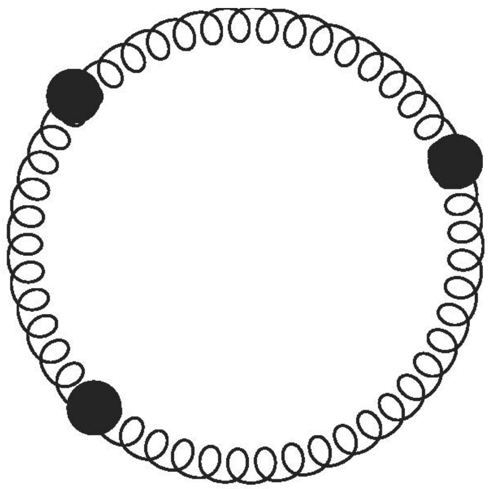
1. Two balls, each of mass $m = 2.00$ kg and negligible radius are attached to a thin rod of length $L = 50.0$ cm of negligible mass. The rod is free to rotate in a vertical plane without friction about a horizontal axis through its center in the gravitational field at the surface of the Earth (assumed to be uniform). With the rod initially stationary and horizontal (see Figure), a wad of wet putty of mass $M = 50.0$ g drops onto one of the balls, hitting it with a speed $v_0 = 3.00$ m/s and then sticking to it.
   (a) What is the angular speed of the system just after the putty wad hits?
   (b) What is the ratio of the kinetic energy of the system after the collision to that of the putty wad just before?
   (c) Through what angle will the system rotate before it momentarily stops?



2. Consider the general problem of N beads of mass m that slide frictionlessly around a circular hoop. Adjacent pairs of beads are connected by springs with identical spring constants and equilibrium lengths. This system of springs and beads forms a closed circle around the hoop. For any given N, the spring constant is chosen such that in equilibrium, the springs are under tension T. Answer the following questions:

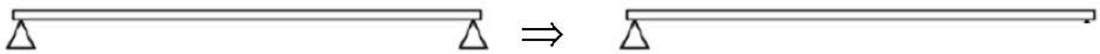
   (a) Suppose $N = 2$. For $t < 0$ bead #1 is held fixed at a reference position, $\theta_1 = 0$, and bead #2 is held fixed at $\theta_2 = \pi + \Delta$ where $\Delta \ll \pi$. At $t = 0$ the beads are released. Find the subsequent motion of the two beads, *i.e.* $\theta_1(t)$ and $\theta_2(t)$.

   (b) Suppose $N = 3$. Find the normal modes of the system and their corresponding frequencies. The figure below shows this three-bead configuration.



3. A mass m can slide without friction in two dimensions on a horizontal surface. It is attached by a massless rope of length L to a second mass M through a hole in the surface. The mass M hangs vertically in a uniform vertical gravitational field g.
   (a) If the mass m moves in a circle of radius r_0 centered on the hole with constant angular velocity ω_0, what is value of ω_0?
   (b) Show that this motion is stable against small perturbations.
   (c) Find the frequency of small oscillations about this circular motion.

4. As shown in the figure, a uniform thin rod of weight W and length L is supported horizontally by two supports, one at each end of the rod. At $t = 0$, one of the supports is removed. Find the force on the remaining support in terms of W immediately thereafter (at $t = 0$). At this instant, what is the angular acceleration around the remaining support in terms of L and g?



5. A block of mass $m = 200$ g is attached to a horizontal spring with spring constant $k = 0.85$ N/m. The other end of the spring is fixed. When in motion, the system is damped by a force proportional to the velocity, with proportionality constant $-b = -0.2$ kg/s.

   (a) Write the differential equation of motion for the system.

   (b) Show that the system is underdamped. Calculate the oscillation period and compare it to the natural period.

   (c) How long does it take for the oscillating block to lose 99.9% of its total mechanical energy? How many cycles does this correspond to? By what factor does the amplitude decrease during this time?

   The system is now subject to a harmonic external force, $F(t) = F_0 \cos(\omega_e t)$, with a fixed amplitude $F_0 = 1.96$ N.

   (d) Calculate the driving frequency $\omega_{e,max}$ at which amplitude resonance (*i.e.* when the amplitude is maximized) occurs, and find the steady-state maximum amplitude to which this corresponds.

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Department of Physics
QUALIFYING EXAMINATION

Monday, January 11, 2016
3:10PM to 5:10PM
Classical Physics
Section 2. Electricity, Magnetism & Electrodynamics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electricity etc.), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

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Good Luck!

1. A current I flows down a long straight wire of radius a. Assume the wire is made of linear material with magnetic susceptibility χ_m, and the current is distributed uniformly across the cross section of the wire.

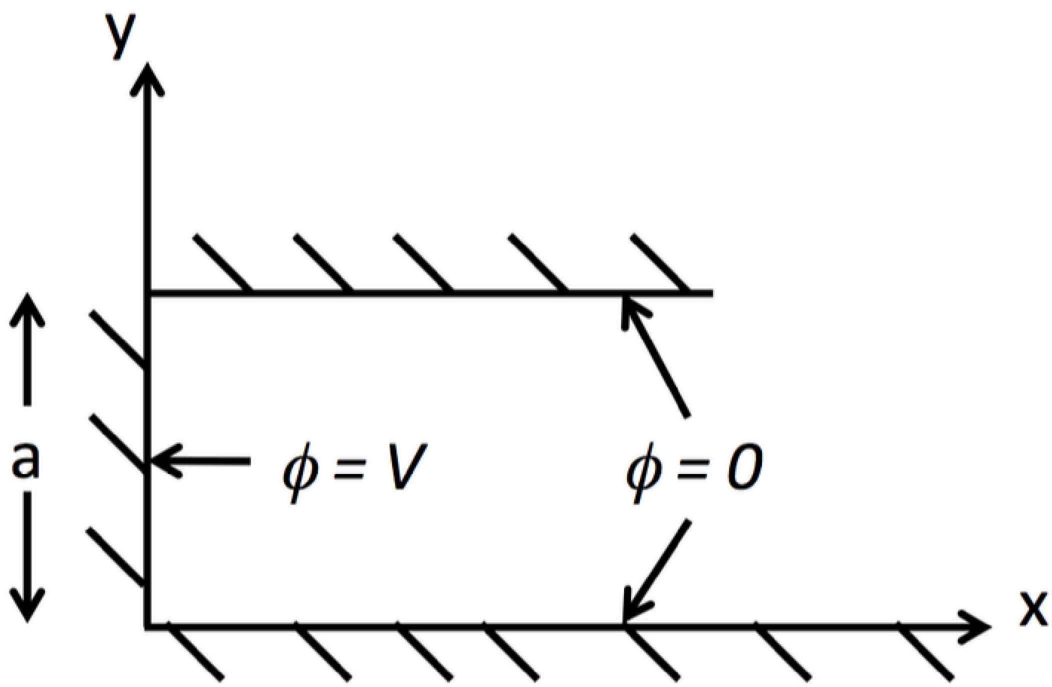
   (a) Calculate the magnetic field a distance s from the axis. Consider separately the regions both inside the wire ($s < a$) and outside ($s > a$).

   (b) Calculate all of the bound currents in the problem. What is the net bound current flowing down the wire?

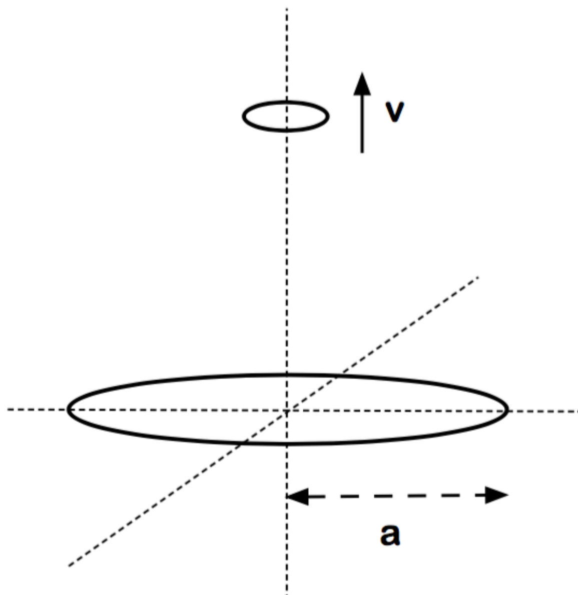
2. An infinitely long 2-D slot has a width a. The walls are conductors held at fixed potentials $\phi = 0$ and $\phi = V$, as shown.

   (a) Determine the electric potential at an arbitrary location (x, y) inside the slot.

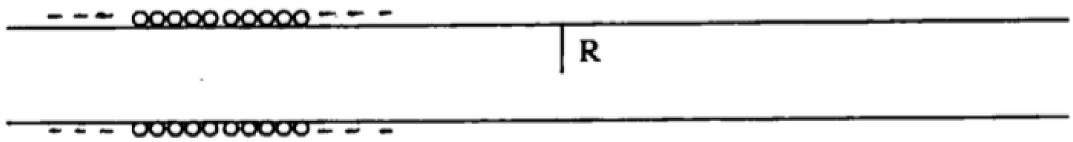
   (b) A positron is released from rest at the coordinates (x,y) = (a, a/2). Find an expression for the force on the positron. Where will it be located at time $t \to \infty$?



3. A thin, circular conducting ring of radius a lies fixed in the $x - y$ plane centered on the z axis. It is driven by a power supply such that it carries a constant current I. Another thin conducting ring of radius b, with $b << a$, and resistance R is centered on and is normal to the z axis. This second ring is moved along the z axis at constant velocity v such that it's center is located at $z = vt$. Estimate, using what ever approximations you consider appropriate, the following quantities including the full time dependence.
   (a) The current in the moving ring.
   (b) The force required to keep the ring moving at constant velocity.



4. A long cylindrical solenoid of radius R and length $L \gg R$ is tightly wound with a single layer of wire (see below). The number of turns per unit length is N/L. The wire breaks when the tension in the wire is greater than T. Find the maximum current that can be carried by the wire.

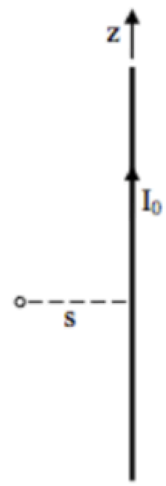


N/L turns/length

5. The general expressions for the scalar and vector potentials are

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t_r)d^3r'}{|\vec{r}-\vec{r}'|}, \quad \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{J(\vec{r}',t_r)d^3r'}{|\vec{r}-\vec{r}'|}, \tag{1}$$

where t_r is the retarded time.



A long (effectively infinite) neutral wire on the z-axis has zero current for $t < 0$. At $t = 0$ a steady current I_0 is suddenly turned on in the $+\hat{z}$ direction (see Figure).

(a) Consider a point at a distance s from the wire ($z = 0$). At what time do the electric and/or magnetic fields first become non-zero at this point? Hereafter call this time t_s ('s' for when the field starts at position s).

(b) What is the value of the scalar potential V at position s at time $t > t_s$?

(c) What is the direction of the vector potential $\vec{A}$ at position s at time $t > t_s$?

(d) What is the direction of the electric field $\vec{E}$ at position s at time $t > t_s$?

(e) What is the direction of the magnetic field $\vec{B}$ at position s at time $t > t_s$?

(f) Write an integral expression for the magnitude of the vector potential $A(s,t)$ at times $t > t_s$.

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Wednesday, January 13, 2016
1:00PM to 3:00PM
Modern Physics
Section 3. Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 3 (QM), Question 2, etc.).

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Good Luck!

1. The ground-state wave function of the hydrogen atom is $\psi_{1s}(r) \sim e^{-r/a_0}$, where the Bohr radius $a_0 = 5.29 \times 10^{-11}$ m. This solution, and the corresponding ground-state energy

$$E_0 = -\frac{e^2}{2a_0} = -13.6 \text{ eV},$$

is derived assuming that the proton is a point charge. In reality the proton's charge is distributed over its radius R, which we take to be 10^{-15} m.

(a) Assuming that the proton's charge is uniformly distributed over a solid sphere of radius R, estimate the fractional shift $\Delta E/|E_0|$ in the ground-state energy of the hydrogen atom from the value obtained assuming the proton is a point charge. [Computing the shift to lowest non-vanishing order in the small parameter R/a_0 suffices.]

(b) How would the order of magnitude of your answer change if the electron in the hydrogen atom were replaced with a negative muon, with mass $m_\mu = 207\, m_e$?

2. Consider a particle in one dimension with a potential energy

$$V(x) = \begin{cases} -U\,, & -c < x < c \\ 0\,, & \text{otherwise} \end{cases}$$

where U is a positive constant.

(a) Consider the wave function

$$\psi(x) = \begin{cases} a(b+x)\,, & -b < x < 0 \\ a(b-x)\,, & 0 < x < b \\ 0\,, & \text{otherwise} \end{cases}$$

where a and b are constants and $b > c$. What is the expectation value of the Hamiltonian in this state?

(b) Use the result from part (a) to show that there will be a bound state for any value of U.

(c) Use the result from part (b) to show that a one-dimensional potential energy that is (i) equal to zero at $x = \pm\infty$ (ii) nowhere greater than zero, and (iii) less than zero in some finite interval, always has a bound state.

3. Consider a particle in one dimension with an *inverted* (upside-down) harmonic oscillator potential $V(x) = -\frac{1}{2}m\omega^2 x^2$. Picking units such that $m = 1$, $\hbar = 1$, the Schrödinger equation takes the form:

$$i\partial_t\psi = -\tfrac{1}{2}\partial_x^2\psi - \tfrac{1}{2}\omega^2 x^2\psi$$

Due to the minus sign in the potential, this describes an unstable system, with energy unbounded from below.

(a) Consider a state at time $t = 0$ given by a Gaussian wave packet of the form

$$\psi_0(x) = \alpha_0\, e^{-\beta_0 x^2}\,, \qquad \beta_0 \equiv \frac{\omega}{2}\tan\theta\,.$$

Here α_0, β_0 and θ are real constants, with the parametrization of β_0 in terms of θ introduced for future convenience. Show that the wave function evolves in time as

$$\psi(x,t) = \alpha(t)\, e^{-\beta(t)\, x^2}\,,$$

and find $\beta(t)$ explicitly.

(b) Find the late time ($t \to \infty$) behavior of $\beta(t)$ and use this to show that at late times, the expectation value of $\hat{x}^2$ is $\langle\hat{x}^2\rangle \propto e^{2\omega t}$. What is the proportionality constant? Here $\hat{x}$ is the position operator.

(c) Show similarly that at late times, the expectation value $\langle(\hat{p}-\omega\hat{x})^2\rangle$ decays exponentially. What is the exponent? Here $\hat{p}$ is the momentum operator. This exponential decay is the signature of a squeezed state.

4. Assume that a particle is moving in an harmonic oscillator potential, $V(x) = \frac{1}{2}m\omega^2 x^2$. At an initial time, say $t = 0$, we are given that its wave function is

$$\psi(x, 0) = N \sum_n \left(\tfrac{1}{\sqrt{7}}\right)^n \psi_n(x)\,,$$

where the $\psi_n(x)$ are the usual orthonormal energy eigenstates of the harmonic oscillator.

(a) Find the value of the normalization constant N.

(b) Show that the probability of finding the particle at a given position x is a periodic function of t, and find the period.

(c) Find the expectation value of the energy.

5. Consider the two-dimensional isotropic harmonic oscillator, picking units such that $m = 1$, $\omega = 1$ and $\hbar = 1$, so the Hamiltonian is

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) \ .$$

with $[x, p_x] = i = [y, p_y]$. Interpreting the system as being made up of two independent one-dimensional oscillators, the energy spectrum can be generated using the standard one-dimensional creation and annihilation operators

$$a_x = \frac{1}{\sqrt{2}}(x + ip_x) \ , \qquad a_x^\dagger \ , \qquad a_y = \frac{1}{\sqrt{2}}(y + ip_y) \ , \qquad a_y^\dagger \ ,$$

in terms of which $H = a_x^\dagger a_x + a_y^\dagger a_y + 1$, and $[H, a_x] = -a_x$, $[H, a_x^\dagger] = a_x^\dagger$, etc.

(a) Construct the energy spectrum using these operators, and find the degeneracy of each energy level.

(b) Consider now the angular momentum operator,

$$L = xp_y - yp_x \ .$$

Express L in terms of the a, $a^\dagger$, and show that the basis of energy eigenstates constructed above does *not* diagonalize the angular momentum operator.

(c) Find a new basis of creation and annihilation operators, obtained as linear combinations of the one-dimensional operators defined above, which generate a basis of energy eigenstates that *does* diagonalize the angular momentum operator. (Hint: You might make use of an analogy with the relation between linear and circular polarization.) What are the possible values of the angular momentum in each energy level?

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Wednesday, January 13, 2016
3:10PM to 5:10PM
Modern Physics
Section 4. Relativity and Applied Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Relativity and Applied QM), Question 2, etc.).

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Good Luck!

1. Consider the first relativistic correction, H_1, in the Hamiltonian for the one dimensional harmonic oscillator.

$$H = H_0 + H_1 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - \frac{1}{2mc^2}\left(\frac{p^2}{2m}\right)^2 \tag{1}$$

Evalulate the first order shift in the ground state energy due to H_1.
Recall that the ground state wave function is given by

$$U_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega x^2}{2\hbar}\right] \tag{2}$$

2. An asteroid is on a collision course with a space station located 5000 light-minutes from Earth. The asteroid is moving away from the Earth toward the space station at a speed of $3/5c$ along a trajectory which is a straight line connecting Earth and the space station. To save the station, NASA launches a missle from Earth at $4/5c$. When the missile is launched, NASA determines that the asteroid is 400 light-minutes from Earth.

   (a) How many minutes should NASA set on a timer located on the missle so that it will explode just as it catches up to the asteroid? (In all parts of this problem, ignore subtleties having to do with acceleration.)

   (b) A few weeks later, another asteroid is on a similar collision course with the space station, travelling again at $3/5c$. NASA decides to send another missile to destroy it, but this time they want the missile to pass the asteroid and explode only when according to sensors on the missile, the missile is 350 light minutes beyond the asteroid. In this scenario, how many minutes should NASA set on the missile's timer?

   (c) From the perspective of the missile's frame of reference, how far away is the asteroid when the missile is launched?

3. A thin plate with surface rest-mass density Σ_0 $[g/cm^2]$ is surrounded by uniform dust at rest with mass density ρ $[g/cm^3]$. At time $t = 0$ the plate is set in motion along its normal with initial Lorentz factor $\gamma(0) = \gamma_0$. The moving plate collides inelastically with the dust particles (which stick to its surface) and gradually decelerates.

   Find the evolution of its Lorentz factor $\gamma(t)$. If $\gamma_0 \gg 1$, at what time $\gamma(t) = \gamma_0/2$?

4. A one-dimensional non-relativistic particle interacts with the potential

$$V(x) = \lambda \frac{\hbar^2}{2m} \delta(x), \tag{3}$$

where λ is a constant and the prefactor is factored out to simplify the algebra.

(a) Calculate the reflection and transmission coefficients (probabilities) as a function of the incident particle wavenumber k.

(b) Calculate the scattering and bound states for $\lambda < 0$. Show that there is a single bound state, and that it is orthogonal to the scattering states.

5. Consider a hydrogen atom. The spin-orbit interaction at radius r is written as:

$$H_{\mathrm{so}} = \frac{e^2}{2m^2c^2r^3}\vec{S}\cdot\vec{L}, \tag{4}$$

where $\vec{S}$ is the spin of the electron and $\vec{L}$.

(a) Describe in words the origin of the spin-orbit interaction.

(b) Construct the basis of wave functions that diagonalize H_{so}.

(c) Obtain the spin-orbit interaction energies for hydrogen in the state with radial quantum number n=2. You may express your answer in terms of the expectation values $\langle 1/r^3\rangle$ of the hydrogen atom states (you do not need to calculate these expectation values explicitly).

Columbia University
Department of Physics
QUALIFYING EXAMINATION

Friday, January 15, 2016
1:00PM to 3:00PM
General Physics (Part I)
Section 5.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 5 (General Physics), Question 2, etc.).

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Good Luck!

1. Suppose the earth's lower atmosphere is approximated as a classical ideal gas with mean molecular weight M in a uniform gravitational field which produces an acceleration g.

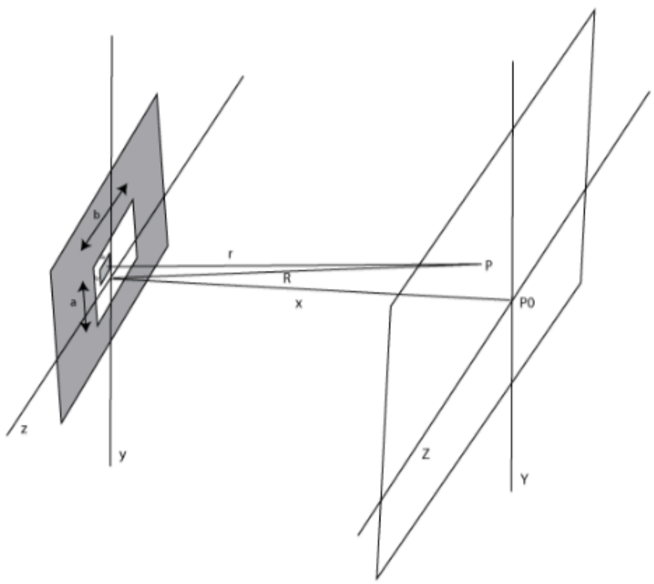
   (a) Assume a small volume of air is in hydrostatic equilibrium with its surrounding atmosphere at temperature T and pressure P. Find an expression for the pressure gradient dP/dz around it in terms of P, T, g and M. (z is the altitude above the earth surface.)

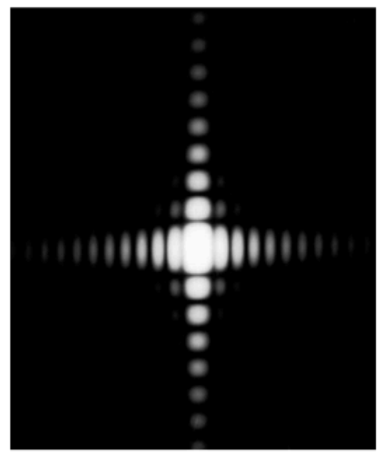
   Our lower atmosphere is very slightly convective. Large blobs of it have zero buoyancy. They move up or move down while adjusting their density to be the same as that of the surrounding air without significant exchange of heat with it. Assume air is composed of diatomic molecules, each of which can be modeled classically as two point masses at the ends of a rigid, massless stick.

   (b) Find an expression for the temperature T at any point in the lower atmosphere in terms of the pressure P at the same point and T_0 and P_0, the temperature and pressure at the surface of the earth respectively, ie., find $T(P, T_0, P_0)$.

   (c) Combine your answers from parts (a) and (b) to determine the temperature and pressure as a function of height z above the earth's surface, ie., find $T(z, T_0)$ and $P(z, P_0, T_0)$.

2. A laser beam is incident from the left (pointed in the $+\hat{x}$ direction) upon a rectangular aperture with length a in the y-direction and length b in the z-direction, where $b > a$ (see Figure). The laser is monochromatic with a wavelength λ and its intensity is uniform across the opening. Light passing through the aperture is collected on a screen at a very large distance $x(\gg a, b, \lambda)$ away from both the aperture and the laser. Coordinates on the distant screen are denoted by their physical coordinates (Y, Z), or by the angles $(\theta_Y \approx Y/x, \theta_Z \approx Z/x)$ measured with respect to the $\hat{x}$ axis.





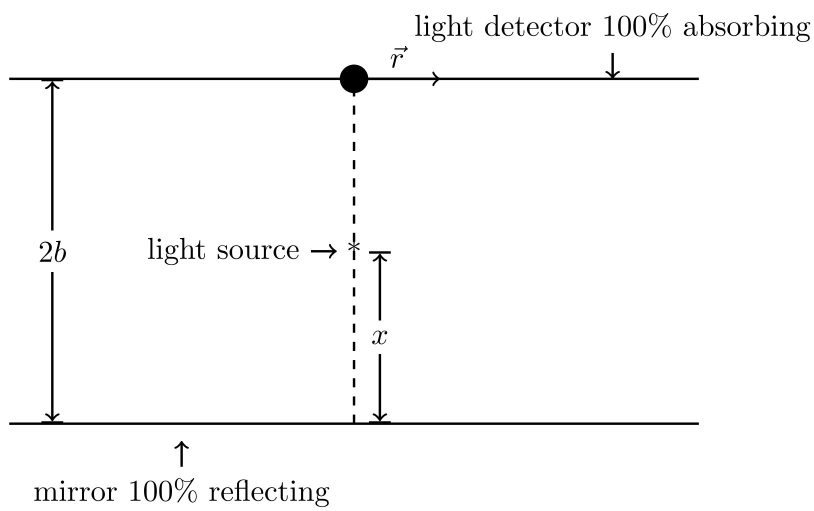
(a) The intensity pattern $I(\theta_Y, \theta_Z)$ as measured on the distant screen is shown in the Figure on the right. Which directions in this figure correspond to the Y axis, and which correspond to the Z axis? (example answer: the vertical direction corresponds to the Y axis) Explain your reasoning.

(b) How does the angular distribution of intensity $I(\theta_Y, \theta_Z)$ change if the screen is moved a factor of 2 further away from the aperture? Explain your answer.

(c) How would the measured intensity distribution $I(\theta_Y, \theta_Z)$ change if the wavelength of the laser light λ is doubled? Briefly explain your answer.

(d) Calculate the intensity distribution $I(\theta_Y, \theta_Z)$ in terms of the angles θ_Y, θ_Z and normalized to the intensity $I(0)$ at the center of the screen. *Hint: Do not worry about the constant amplitude in front of the electric field or intensity until the end of the derivation.*

3. A point dipole source of monochromatic light (wavelength λ) is suspended a height x above a perfectly reflecting mirror. Emitted light directed down towards the mirror below has the same amplitude and polarization as light emitted upwards in the opposite direction. A planar, fully-absorbing light detector, oriented exactly parallel to the mirror is placed a height $2b$ above the mirror. The detected light intensity depends on the source position x and on r, the distance from an axis through the source and normal to the mirror and detector, as shown in the figure. Assume the light emitter is small enough that it intercepts none of the reflected light. Assume in what follows that $\lambda \ll b$ and $r \ll 2b - x$. There are many local maxima at $r = 0$ as x is varied, and at any fixed x as r is varied.



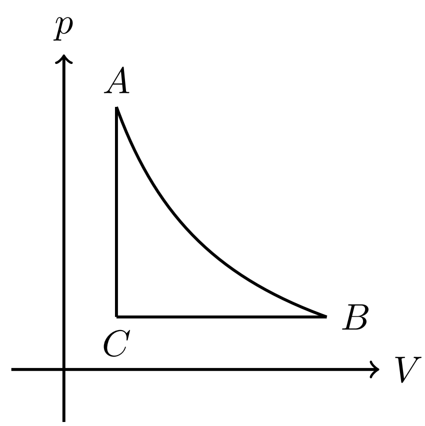
(a) At what values of x are intensity maxima observed on the light detector at $r = 0$?

(b) The source is placed at $x = b$ and the wavelength λ is such that an intensity maximum is observed on the light detector at $r = 0$. At what other values of r will intensity maxima be observed on the light detector?

(c) How is the answer to part (a) changed if the mirror moves with velocity v parallel to the detector plane? How is the answer to part (b) changed?

4. One mole of a diatomic ideal gas is driven around the cycle ABCA shown on the pV diagram below. Step AB is isothermic, with a temperature $T_A = 500\ K$; step BC is isobaric; and step CA is isochoric. The volume of the gas at point A is $V_A = 1.00\ L$, and at point B is $V_B = 4.00\ L$. Treat a diatomic gas molecule as two point masses at the ends of a rigid, massless rod.

   The ideal gas constant is $R = 8.31\ J/mol \cdot K$.



   (a) What is the pressure p_B at point B?

   (b) What is the total work done in completing one cycle (ABCA)?

   (c) What is the entropy change $S_c - S_B$?

5. A rope of uniform linear density μ and total length L is suspended from one end and hangs vertically under its own weight. It is lightly tapped at the lower end.

   How long does it take for the perturbation to reach the top of the rope?

6. For a non-relativistic ideal gas the partition function for N particles (of mass m) in a volume V is

$$Z_N = \frac{Z_1^N}{N!} \tag{1}$$

where

$$Z_1 = \int \frac{d^3k d^3p}{h^3} \exp[-\frac{\vec{p}^2}{2mkT}] = \frac{V}{\lambda^3} \tag{2}$$

with

$$\lambda = \frac{h}{\sqrt{2\pi mkT}} \tag{3}$$

Now consider an extreme-relativistic ideal gas of N particles in a volume V. Give the following:

(a) The partition function Z_N at a temperature T

(b) $E(N, T)$ the energy of the gas

(c) P the pressure of the gas

If m is the mass of the particle when do you expect the extreme-relativistic approximation to be good?

Columbia University
Department of Physics
QUALIFYING EXAMINATION

Friday, January 15, 2016
3:10PM to 5:10PM
General Physics (Part II)
Section 6.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 6 (General Physics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

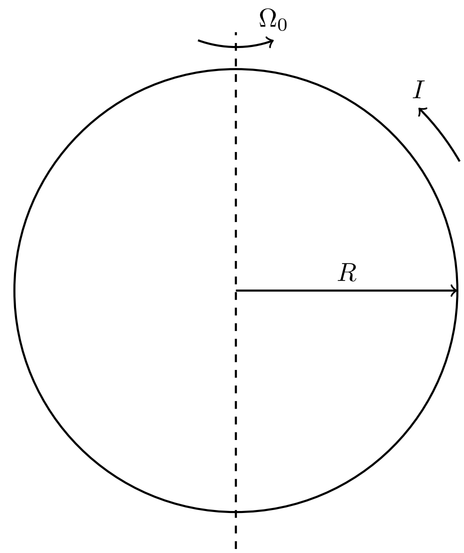
You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Electric current I is circulating along an ideally conducting thin hoop of mass M and radius R. The hoop is placed in vacuum and infinite volume and constrained to rotate on an axis that passes through the conductor and center of the hoop as shown in the figure. Initially the hoop rotates about this axis with angular velocity $\vec{\Omega}_0$. How will its angular velocity evolve with time? [Partial credit will be given for a solution based on dimensional analysis.]



2. In high energy nuclear collisions between nucleus A and B, it is conventional to specify the center-of-mass collision energy in terms of $\sqrt{S_{NN}}$, which is the energy of the collision between one nucleon from A and one nucleon from B, assuming that both nucleons are motionless in their parent nucleus's rest frame. (Here nucleon denotes either a proton or a neutron.)

   In reality the nucleons are not motionless when viewed from the rest frame of the nucleus, due to their Fermi momentum. The density of nucleons in a large nucleus is about 0.16 fm^3, where $1 fm = 10^{-15} m$.

   (a) Find the Fermi momentum p_F for a nucleus with the above density, assuming zero temperature and an equal density of protons and neutrons.
   (b) At RHIC, the nominal value of $\sqrt{S_{NN}}$ is 200 GeV when colliding beams of nuclei having equal energies but opposite directions. Find the range of energies about this central value due to Fermi momenta within each nucleus aligned and anti-aligned with the collision direction.
   (c) Repeat part (b) for the LHC, where $\sqrt{S_{NN}} = 5000\, GeV$.

   For this problem, you can take the rest energy of a nucleon to be $M_N c^2 \approx 1000\, GeV$. You may find it convenient to use $\hbar c \approx 0.2\, GeV \cdot fm$.

3.

(a) An experiment requires a flat electrode surface to stay free of adsorbed molecules for a duration τ (the maximum allowed adsorption coverage is $f < 10\%$). Assuming that each incident molecule sticks to the surface, estimate the maximum allowed background air pressure P in terms of τ, f, the temperature T, and the typical mass M and diatemeter d of an adsorbed molecule.

(b) Estimate the order-of-magnitude numerical value of P from part (a) at room temperature, for $\tau \sim 1$ hour.

4.

(a) In solid metals the effective mass of a conduction electron can be different from that of a bare mass electron. Explain the concept of effective mass, and describe why the effective and bare electron masses may be different.

(b) Describe an experiment to determine the effective mass of an electron in metals. It can either be a direct experiment or a combination of a few measurements of other properties which allows a derivation of the effective electron mass

(c) Suppose you had access to a beam of neutrons at a neutron scattering facility, and all its relevant equipment. Explain how you would go about measuring the mass of the neutron.

5. We have an ideal gas composed of N He atoms contained in a vessel of volume V. The vessel is a cube of volume $V = L^3$, where L is the length of the cube. Consider the limit of low temperature ($T \to 0$) and assume that the system is an ideal gas at all temperatures.

   (a) Consider cyclic boundary conditions for the wavefunctions of momentum and energy to obtain the energy states E. [In cyclic boundary conditions the wave functions can be viewed as defined in an infinite volume, but are required to be unchanged by translation through a distance L in the x or y directions]. Calculate the density of states as a function of energy.

   (b) The He^3 isotope has two protons and one neutron in its nucleus. He^3 atoms have spin one-half.

   (i) What is the value of the chemical potential μ at $T = 0$?

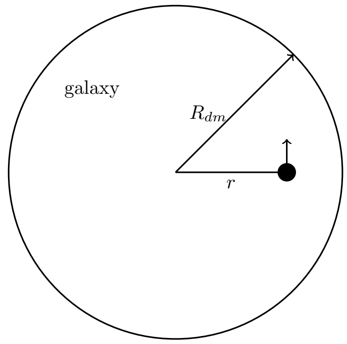
   (ii) Assume that the temperature is raised slightly so that T remains small. The total energy of the He^3 ideal gas is written as $U(T) = U(0) + F(T)$. Use phenomenology (qualitative) considerations to show that the leading term in $F(T)$ is proportional to T^2.

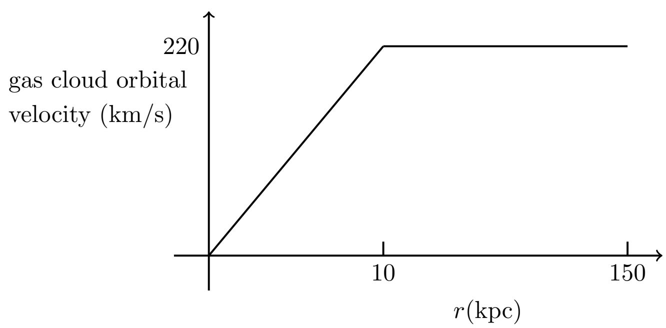
   (c) The He^4 isotope has two protons and two neutrons in its nucleus. He^4 atoms have zero spin.

   (iii) Show that the chemical potential μ is negative.

   (iv) Obtain an expression for the temperature at which the value of the chemical potential comes very close to zero ($\mu \to 0$).

6. Assume a toy model for a spherical galaxy in which the mass of the dark matter is much larger than that of the visible matter $M_{dm} \gg M_{visible}$. Assume the dark matter is spherically symmetrically distributed (with unknown density distribution) in a sphere of radius $R_{dm} = 150\ kpc$ ($1\ kpc = 3 \times 10^{19}\ m$). Gas clouds are observed to orbit inside the galaxy at various radii $r < R_{dm}$. The orbital velocity of these gas clouds is observed to be roughly constant for $r \gtrsim 10\ kpc$ with $v \sim 220\ km/s$, as shown by the solid line in the second figure.





(a) Use the observation of constant gas cloud orbital velocities at $r \gtrsim 10\ kpc$ to deduce the density distribution of dark matter as a function of radius $\rho(r)$.

(b) Estimate the total amount of dark matter in this galaxy. Express your answer in solar masses ($1\ M_{sol} = 2 \times 10^{30}\ kg$).

(c) The visible matter in this galaxy, $M_{visible} \sim 5 \times 10^{11}\ M_{sol}$, is concentrated at $r \lesssim 10\ kpc$. Explain qualitatively how this accounts for the rotation curve behavior at $r \lesssim 10\ kpc$.

(d) It is believed that the dark matter in galaxies cannot be dominantly composed of particles of the Standard Model. Why?